Thermodynamic Properties of the Two-Dimensional Two-Component Plasma

L. Šamaj[1] **and I. Travěnec**[1]

The model under consideration is a two-dimensional two-component plasma, stable against collapse for the dimensionless coupling constant $\beta < 2$. The combination of a technique of renormalized Mayer expansion with the mapping onto the sine-Gordon theory provides the full thermodynamics of the plasma in the whole stability range of β. The explicit forms of the density–fugacity relationship and of the specific heat (at constant volume) per particle are presented.

1. INTRODUCTION

The model under consideration is the two-dimensional (2D) two-component plasma (TCP), i.e., a neutral system of pointlike particles of positive and negative charge, interacting through the 2D logarithmic Coulomb interaction. Classical equilibrium statistical mechanics is applied. The system is stable against collapse for the dimensionless coupling constant β (sometimes denoted as Γ) lower than 2. The absence of some short-range interaction, e.g., the hard-core one, prevents to extend the treatment beyond $\beta = 2$ and to reach the Kosterlitz–Thouless phase transition which occurs around $\beta = 4$.

In arbitrary dimension, the long-range tail of the Coulomb potential causes screening, and thus gives rise to exact constraints for the charge-charge structure function (see review[1]). In the bulk regime, the zeroth- and second-moment conditions for the structure function are known exactly.[2, 3] The specialization to the 2D TCP brings additional sum rules.

---

[1] Institute of Physics, Slovak Academy of Sciences, Dúbravská cesta 9, 842 28 Bratislava, Slovakia; e-mail: fyzimaes@savba.sk and fyzitrav@savba.sk

The knowledge of the exact equation of state[4] fixes via the compressibility equation the zeroth moment of the *number density* correlation function.[5] In a recent paper,[6] the second-moment formula for that density–density correlation function has been derived by using analogies with critical systems. This result was rederived in ref. 7 directly by using a renormalized Mayer expansion,[8, 9] having some common features with Salpeter's expansion procedure.[10]

At arbitrary temperature, the 2D TCP is supposed to be in critical state in terms of the induced electrical-field correlations (but not the particle correlations).[11, 12] The free energy therefore exhibits a universal finite-size correction predicted by the Conformal Invariance theory.

The 2D TCP is mappable onto the Quantum Field Theory (QFT) models, namely the 2D Thirring model (see e.g., ref. 13) and the 2D Euclidean, or $(1+1)$-dimensional quantum, sine-Gordon model.[14–16] The integrability of the sine-Gordon theory was established and analysed in refs. 17–21 and in many others.

This paper is devoted to the derivation of the thermodynamics of the 2D TCP in the whole stability interval $0 \leqslant \beta < 2$. The aim is reached via a study of the relationship between the fugacity z and the total number density of particles n. It is known from a scaling argument that the combination $n^{(1-\beta/4)}/z$ is a function of β only. Up to now, the exact form of the $n-z$ relationship was known only in the Debye–Hückel $\beta \to 0$ limit, $n/z \to 2$, and at the exactly solvable collapse border $\beta = 2$,[22, 23] when $\sqrt{n}/z \to \infty$. The little interest in the topic in the past is surprising in view of the fact that in the QF theories related to the 2D TCP one-point expectations of local fields are the objects of primary importance. In the Conformal Perturbation treatment of a QFT, all multiple correlation functions are reducible by the Operator-Product-Expansion[24] to the one-point vacuum expectations which contain all the non-perturbative information about the QFT.

We first investigate the $n-z$ relationship by using the renormalized Mayer expansion for many-component fluids[7] and find out that $n^{(1-\beta/4)}/z \sim \beta^{\beta/4} \times$ an analytic function of solely β, $f(\beta)$, whose Taylor expansion around $\beta = 0$ can be constructed systematically by evaluating certain convergent integrals of Bessel functions. The mapping onto the 2D sine-Gordon theory, whose normalization of the relevant cos -field is consistent with the formalism of the renormalized Mayer expansion, then provides the explicit form of the function $f(\beta)$, checked on a few lower orders of its β-expansion. The exact density–fugacity relationship provides the full thermodynamics of the 2D TCP.

The paper is organized as follows. Section 2 recapitulates briefly the ordinary Mayer formalism for many-component fluids. The procedure of the renormalization of the Mayer expansion is described in Section 3. The

application to the 2D TCP is presented in Section 4. The information gained from the mapping onto the 2D sine-Gordon QFT is the subject of Section 5. The explicit form of the specific heat (at constant volume) per particle is given together with a discussion of the results in Section 6.

2. ORDINARY MAYER EXPANSION FOR MANY-COMPONENT FLUIDS

We first consider a general classical multi-component fluid in thermodynamic equilibrium at temperature T. The last system is composed of distinct species of particles $\{\sigma\}$. The particles may be exposed to an external potential $u(\mathbf{r}, \sigma)$ and interact with each other through the pair potential $v(\mathbf{r}_i, \sigma_i \mid \mathbf{r}_j, \sigma_j)$ which depends on the mutual distance $|\mathbf{r}_i - \mathbf{r}_j|$ of particles i, j as well as on their types σ_i, σ_j. Vector position $\mathbf{r}_i$ of a particle in d-dimensional space will be sometimes represented simply by i. The grand-canonical partition function is defined by

$$\Xi(T, V, \{z_\sigma\})$$
$$= \sum_{\{N_\sigma\}} \prod_\sigma \frac{1}{N_\sigma!} \int_V \prod_{i=1}^N [\,\mathrm{d}i\, z(i, \sigma_i)\,] \exp\left[-\frac{\beta}{2} \sum_{(i \neq j) = 1}^N v(i, \sigma_i \mid j, \sigma_j) \right] \quad (1)$$

where the first sum runs over all possible species numbers, $N = \sum_\sigma N_\sigma$ is the total particle number, $\beta = 1/kT$ is the inverse temperature and $z(\mathbf{r}, \sigma)$ of dimension $[\text{length}]^{-2}$ denotes the position-dependent fugacity of σ-particles. In the direct format, i.e., with fugacities $\{z(\mathbf{r}, \sigma)\}$ as controlling variables, Ξ is the generator for the particle densities $\{n(\mathbf{r}, \sigma)\}$ in the sense that

$$n(\mathbf{r}, \sigma) = z(\mathbf{r}, \sigma) \frac{\delta \ln \Xi[z]}{\delta z(\mathbf{r}, \sigma)} \quad (2)$$

The transition to the inverse format, i.e., with densities $\{n(\mathbf{r}, \sigma)\}$ as controlling variables, is based on the Legendre transformation

$$-\beta \bar{F}[n] = \ln \Xi - \int_V \mathrm{d}\mathbf{r} \sum_\sigma n(\mathbf{r}, \sigma) \ln z(\mathbf{r}, \sigma) \quad (3)$$

which defines the Helmholtz free energy $\bar{F}$ as the explicit density functional. The subtraction of the one-particle part provides the dimensionless (for notational convenience, minus) excess free energy

$$\Delta[n] = -\beta \bar{F}[n] + \int_V \mathrm{d}\mathbf{r} \sum_\sigma [\,n(\mathbf{r}, \sigma) \ln n(\mathbf{r}, \sigma) - n(\mathbf{r}, \sigma)\,] \quad (4)$$

It is easy to show that $\Delta[n]$ is the generator in the following sense

$$\ln\left[\frac{n(\mathbf{r}, \sigma)}{z(\mathbf{r}, \sigma)}\right] = \frac{\delta\Delta[n]}{\delta n(\mathbf{r}, \sigma)} \tag{5}$$

Its Mayer diagrammatic representation in density reads[25]

$\Delta[n] = \{$ all connected diagrams which consist of $N \geqslant 2$ field $n(i, \sigma_i)$-circles

and $f(i, \sigma_i \mid j, \sigma_j)$-bonds, and are free of connecting circles$\}$ (6)

(the removal of a connecting circle disconnects the diagram). Here,

$$f(i, \sigma_i \mid j, \sigma_j) = \exp[-\beta v(i, \sigma_i \mid j, \sigma_j)] - 1 \tag{7}$$

is called the Mayer function and, besides the integration over spatial coordinate of a field (black) circle, the summation over all σ-states at this vertex is assumed as well.

3. RENORMALIZED MAYER EXPANSION

The renormalized Mayer representation of $\Delta[n]$ results from Eq. (6) in two steps:[8, 9]

(i) the expansion of each Mayer function in the inverse temperature,

$$f(1, \sigma_1 \mid 2, \sigma_2) = -\beta v(1, \sigma_1 \mid 2, \sigma_2) + \frac{1}{2!}[-\beta v(1, \sigma_1 \mid 2, \sigma_2)]^2 + \cdots \tag{8}$$

or, graphically,

$$\underset{1,\sigma_1 \qquad 2,\sigma_2}{\circ\!\!-\!\!\overset{f}{-}\!\!-\!\!\circ} = \underset{1,\sigma_1 \qquad 2,\sigma_2}{\circ\!\!-\!\!\overset{-\beta v}{-}\!\!-\!\!\circ} + \underset{1,\sigma_1 \qquad\qquad 2,\sigma_2}{\propto\,\overset{\frown}{\underset{\smile}{}}\,\infty} + \cdots \tag{8'}$$

where the factor 1/(number of interaction lines)! is automatically assumed;

(ii) the consequent series elimination of two-coordinated field circles between every couple of three- or more-coordinated field circles (hereinafter, by coordination of a circle we mean its bond-coordination, i.e., the number of bonds meeting at this circle). The renormalized K-bonds are given by

$$\underset{1,\sigma_1 \qquad\qquad 2,\sigma_2}{\overset{K}{\circ\!\!\wedge\!\!\wedge\!\!\wedge\!\!\wedge\!\!\circ}} = \underset{1,\sigma_1 \qquad 2,\sigma_2}{\circ\!-\!-\!-\!\circ} + \underset{1,\sigma_1 \qquad\qquad 2,\sigma_2}{\circ\!-\!-\!-\!\bullet\!-\!-\!-\!\circ} + \cdots \tag{9}$$

or, algebraically,

$$K(1, \sigma_1 \mid 2, \sigma_2) = [-\beta v(1, \sigma_1 \mid 2, \sigma_2)] + \sum_{\sigma_3} \int_V \mathrm{d}3 [-\beta v(1, \sigma_1 \mid 3, \sigma_3)]$$
$$\times\, n(3, \sigma_3)\, K(3, \sigma_3 \mid 2, \sigma_2) \qquad (9')$$

The procedure of bond-renormalization transforms the ordinary Mayer representation (6) of Δ into[7]

$$\Delta[n] = \bullet\text{---}\bullet + D_0[n] + \sum_{s=1}^{\infty} D_s[n] \qquad (10a)$$

where

$$D_0 = \bullet\underset{\smile}{\frown}\bullet + \overset{\bullet}{\diagup\diagdown} + \begin{array}{c}\bullet\text{--}\bullet\\ \vdots\quad\vdots\\ \bullet\text{--}\bullet\end{array} + \cdots$$

$$= \sum_{N=2}^{\infty} \frac{1}{2N} \sum_{\sigma_1 \cdots \sigma_N} \int_V \prod_{i=1}^{N} [\mathrm{d}i\, n(i, \sigma_i)][-\beta v(1, \sigma_1 \mid 2, \sigma_2)]$$
$$\times [-\beta v(2, \sigma_2 \mid 3, \sigma_3)] \cdots [-\beta v(N, \sigma_N \mid 1, \sigma_1)] \qquad (10b)$$

is the sum of all unrenormalized ring diagrams (which cannot undertake the renormalization procedure because of the absence of three- or more-coordinated field points) and

$$\sum_{s=1}^{\infty} D_s = \{\text{all connected diagrams which consist of } N \geqslant 2 \text{ field}$$

$n(i, \sigma_i)$-circles of (bond) coordination $\geqslant 3$ and multiple

$K(i, \sigma_i \mid j, \sigma_j)$-bonds, and are free of connecting circles $\}$ (10c)

represents the set of all remaining completely renormalized graphs. By multiple K-bonds one means the possibility of an arbitrary number of K-bonds between a couple of field circles, with the obvious topological factor $1/(\text{number of bonds})!$. The order of s-enumeration is irrelevant, let us say

$$D_1 \qquad D_2 \qquad D_3 \qquad D_4 \qquad D_5 \qquad D_6$$

$$(11)$$

etc.
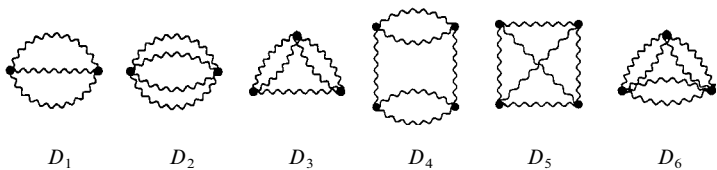
In accordance with relation (5), $\ln[n(1, \sigma_1)/z(1, \sigma_1)]$ is expressible in the renormalized format as follows

$$\ln\left[\frac{n(1, \sigma_1)}{z(1, \sigma_1)}\right] = \underset{1, \sigma_1}{\circ} - - -\bullet + d_0(1, \sigma_1) + \sum_{s=1}^{\infty} d_s(1, \sigma_1) \qquad (12a)$$

where $d_0(1, \sigma_1) = \delta D_0/\delta n(1, \sigma_1)$ can be readily obtained in the form

$$d_0(1, \sigma_1) = \tfrac{1}{2} \lim_{2 \to 1} \left[K(1, \sigma_1 \mid 2, \sigma_2) + \beta v(1, \sigma_1 \mid 2, \sigma_2) \right]|_{\sigma_2 = \sigma_1} \qquad (12b)$$

and

$$d_s(1, \sigma_1) = \frac{\delta D_s}{\delta n(1, \sigma_1)} \qquad (12c)$$

$(s = 1, 2,...)$ denotes the whole family of $(1, \sigma_1)$-rooted diagrams generated from D_s. To get a family, one has to consider the functional dependence of the dressed K-bonds (9) on the species densities as well. Since relation (9′) yields

$$\frac{\delta K(1, \sigma_1 \mid 2, \sigma_2)}{\delta n(3, \sigma_3)} = K(1, \sigma_1 \mid 3, \sigma_3) \, K(3, \sigma_3 \mid 2, \sigma_2) \qquad (13)$$

the root (white) circle is generated, besides the field-circle positions, also on K-bonds, causing their correct K-K division. For example, in the case of generator D_1 drawn in (11), one gets

$$d_1(1, \sigma_1) = \underset{1, \sigma_1}{\underbrace{}} + \overset{1, \sigma_1}{\underbrace{}} \qquad (14)$$

4. THE 2D TCP

Let us now concentrate on the neutral 2D TCP of positive $(\sigma = +)$ and negative $(\sigma = -)$ pointlike unit $(q = 1)$ charges, with the Coulomb interaction energy given by

$$-\beta v(i, \sigma_i \mid j, \sigma_j) = \sigma_i \sigma_j [-\beta v(i, j)] \qquad (15a)$$

$$v(i, j) = -\ln(|i - j|/r_0) \qquad (15b)$$

where the length constant r_0, which fixes the zero point of energy, is for simplicity set to unity and β now plays the role of the dimensionless coupling constant. Considering the regime with homogeneous densities $n(\mathbf{r}, \sigma) = n_\sigma$, the requirement of the charge neutrality implies $n_+ = n_- = n/2$ (n is the total number density of particles). In the fugacity language, the chemical potentials of $\pm$ charged particles must equal to one another, $z(\mathbf{r}, \sigma) = z(\mathbf{r})$; in the infinite-volume limit, $z(\mathbf{r})$ acquires its bulk value z.

The renormalized bonds, defined by (9'), now take the form

$$K(i, \sigma_i \mid j, \sigma_j) = \sigma_i \sigma_j K(i, j) \tag{16a}$$

where $K(i, j)$ satisfy the relation

$$K(1, 2) = [-\beta v(1, 2)] + \int_V \mathrm{d}3 [-\beta v(1, 3)] \, n \, K(3, 2) \tag{16b}$$

In the $V \to \infty$ limit, characterized by translationally invariant $K(i, j) = K(|i - j|)$, the Fourier transformation of (16b) results in

$$\hat{K}(k) = [-\beta \hat{v}(k)] + 2\pi n [-\beta \hat{v}(k)] \, \hat{K}(k) \tag{17}$$

Since the Fourier component $\hat{v}(k) = 1/k^2$, one arrives at

$$K(r) = -\beta \int \frac{\mathrm{d}^2 k}{2\pi} \frac{1}{k^2 + 2\pi\beta n} \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r})$$

$$= -\beta K_0(r \sqrt{2\pi\beta n}) \tag{18}$$

where K_0 is the modified Bessel function of second kind. The special scaling form of K has a fundamental impact on the n- and β-classification of renormalized diagrams.

Let us first study the renormalized representation (10) of the generator $\Delta(n)$:

(i) The first term on the rhs of (10a)

$$\frac{1}{2!} \sum_{\sigma_1, \sigma_2 = \pm 1} \int_V \mathrm{d}1\mathrm{d}2 \, n(1, \sigma_1) \, \sigma_1 \sigma_2 [-\beta v(1, 2)] \, n(2, \sigma_2) \tag{19}$$

is fixed to zero by the charge neutrality.

(ii) The second term D_0 (10b) is expressible as follows

$$D_0(n) = \sum_{N=2}^{\infty} \frac{n^N}{2N} \int_V \prod_{i=1}^{N} \mathrm{d}i [-\beta v(1, 2)][-\beta v(2, 3)] \cdots [-\beta v(N, 1)] \tag{20}$$

or, equivalently,

$$D_0(n) = \frac{1}{2} \int_0^n dn' \sum_{N=2}^\infty n'^{(N-1)} \int_V \prod_{i=1}^N di\, [-\beta v(1,2)][-\beta v(2,3)] \cdots [-\beta v(N,1)] \tag{20'}$$

In the $V \to \infty$ limit, with respect to (16b), the sum over N on the rhs of (20') is nothing but $V \times \lim_{r \to 0} [K(r) + \beta v(r)]$ evaluated at $n = n'$. Using the small-x expansion of $K_0(x)$,[26]

$$K_0(x) = -\ln\left(\frac{x}{2}\right) I_0(x) + \sum_{i=0}^\infty \frac{x^{2i}}{2^{2i}(i!)^2} \psi(i+1) \tag{21}$$

where

$$I_0(x) = \sum_{i=0}^\infty \frac{x^{2i}}{2^{2i}(i!)^2} \qquad \text{and} \qquad \psi(x) = \frac{d}{dx} \ln \Gamma(x)$$

is the psi function $[\psi(1) = -C,\ C$ the Euler's constant$]$, one finally gets

$$\frac{D_0(n)}{V} = \frac{\beta}{4}(n \ln n - n) + \frac{\beta n}{2}\left[C + \frac{1}{2} \ln\left(\frac{\pi \beta}{2}\right) \right] \tag{22}$$

(iii) Let the given completely renormalized diagram D_s ($s = 1, 2,...$), belonging to the sum (10c), be composed of N_s skeleton vertices $i = 1,..., N_s$ of coordination $v_i \geq 3$ and L_s bonds $\alpha = 1,..., L_s$. D_s can be formally expressed as

$$D_s[n] = t_s \sum_{\sigma_1 \cdots \sigma_{N_s} = \pm 1} \int_V \prod_{i=1}^{N_s} [di\, n(i, \sigma_i)] \prod_{\alpha=1}^{L_s} K(\alpha_1, \sigma_{\alpha_1} \mid \alpha_2, \sigma_{\alpha_2}) \tag{23}$$

where t_s is the topological factor and $\alpha_1, \alpha_2 \in \{1,..., N_s\}$, $\alpha_1 < \alpha_2$, denotes the ordered pair of vertices joint by the α-bond. Since, according to (16a), $K(\alpha_1, \sigma_{\alpha_1} \mid \alpha_2, \sigma_{\alpha_2}) = \sigma_{\alpha_1} \sigma_{\alpha_2} K_\alpha$, $K_\alpha \equiv K(\alpha_1, \alpha_2)$, Eq. (23) can be rewritten as follows

$$D_s[n] = t_s \sum_{\sigma_1 \cdots \sigma_{N_s} = \pm 1} \int_V \prod_{i=1}^{N_s} [di\, n(i, \sigma_i)\, \sigma_i^{v_i}] \prod_{\alpha=1}^{L_s} K_\alpha \tag{24}$$

The set of coordination numbers $\{v_i\}$ is constrained by $\sum_{i=1}^{N_s} v_i = 2L_s$ as every bond is shared by just two vertices. For $n(i, \sigma_i) = n/2$, one has

$$D_s(n) = t_s \left(\frac{n}{2}\right)^{N_s} \sum_{\sigma_1 \cdots \sigma_{N_s} = \pm 1} \prod_{i=1}^{N_s} \sigma_i^{v_i} \int_V \prod_{i=1}^{N_s} di \prod_{\alpha=1}^{L_s} K_\alpha \tag{25}$$

i.e., $D_s(n) \neq 0$ if and only if the coordinations of all vertices $\{v_i\}$ in D_s are even numbers ($\geqslant 4$). Let us suppose that this condition is fulfilled. In the $V \to \infty$ limit, due to the invariance of the integrated product $\prod_{\alpha=1}^{L_s} K_\alpha$ with respect to a uniform shift in all integration variables $\{i\}$, one of these variables can be chosen as a reference put at the origin $\mathbf{0}$, with the simultaneous multiplication by volume V,

$$D_s(n) = t_s n^{N_s} V \int \prod_{i=1}^{N_s} \mathrm{d}i\, \delta(j) \prod_{\alpha=1}^{L_s} K_\alpha \qquad (26)$$

$j = 1,\ldots, N_s$. The scaling form of $K_\alpha = -\beta K_0(|\alpha| \sqrt{2\pi\beta n})$ permits us to perform the n- and β-classification of the integral in (26). Every dressed bond K_α brings the factor $-\beta$ and enforces the substitution $r' = r\sqrt{2\pi\beta n}$ which manifests itself as the factor $1/(2\pi\beta n)$ for each field-circle integration $\sim \int r\, \mathrm{d}r$. Since there are $(N_s - 1)$ independent field-circle integrations in (26) we conclude that

$$\frac{D_s(n)}{V} = n\beta^{L_s - N_s + 1} d_s \qquad (27a)$$

where d_s is an intensive quantity, the number

$$d_s = \frac{D_s(n=1, \beta=1)}{V} \qquad (27b)$$

The first nonzero diagram from the sketch (11) is D_2. It contributes to the β^3 order, with

$$d_2 = \frac{1}{2!\,4!} \int \frac{\mathrm{d}^2 r}{2\pi} K_0^4(\mathbf{r})$$

$$= \frac{1}{2!\,4!} \frac{7}{8} \zeta(3) \qquad (28)$$

where ζ is the Riemann's zeta function (see Appendix). In the next β^4 order only diagram D_6 survives, and

$$d_6 = \frac{1}{3!(2!)^3} \int \frac{\mathrm{d}^2 r_1}{2\pi} \int \frac{\mathrm{d}^2 r_2}{2\pi} K_0^2(\mathbf{r}_1)\, K_0^2(\mathbf{r}_2)\, K_0^2(\mathbf{r}_1 - \mathbf{r}_2)$$

$$= \frac{1}{3!(2!)^3} \frac{3}{16} \zeta(3) \qquad (29)$$

(see Appendix), etc.

The above paragraphs (i)–(iii) are summarized by formula

$$\frac{\Delta(n)}{V} = \frac{\beta}{4}(n \ln n - n) + \frac{\beta n}{2}\left[C + \frac{1}{2}\ln\left(\frac{\pi\beta}{2}\right)\right] + n \sum_{s=1}^{\infty} d_s \beta^{L_s - N_s + 1} \qquad (30)$$

Here, $\{d_s\}$ are the numbers yielded by the topology of the corresponding diagrams $\{D_s\}$, nonzero only if the coordinations of all vertices are even numbers $\geqslant 4$.

In order to evaluate $\ln(n_\sigma/z_\sigma)$ using (5), we first recall the well-known formula

$$\frac{\partial \Delta(n)}{\partial n} = \sum_\sigma \int d\boldsymbol{r}\, \frac{\delta \Delta[n]}{\delta n(\boldsymbol{r}, \sigma)}\, \frac{\partial n(\boldsymbol{r}, \sigma)}{\partial n} \qquad (31)$$

valid for an arbitrary functional $\Delta[n]$ with $n(\boldsymbol{r}, \sigma)$ substituted by some function of n. In the considered case $n(\boldsymbol{r}, +) = n(\boldsymbol{r}, -) = n/2$, relation (31) takes the form

$$\frac{\partial \Delta(n)}{\partial n} = \frac{V}{2}\left(\left.\frac{\delta \Delta[n]}{\delta n(\boldsymbol{r}, +)}\right|_{n(\boldsymbol{r}, \sigma) = n/2} + \left.\frac{\delta \Delta[n]}{\delta n(\boldsymbol{r}, -)}\right|_{n(\boldsymbol{r}, \sigma) = n/2} \right) \qquad (32)$$

where the reference $\boldsymbol{r}$-independence of the functional derivatives was assumed. For diagrams in $\Delta[n]$, the direct link between the $+$ and $-$ states of the root point is realized through the state $\{\sigma_i \to -\sigma_i\}$ transformation at field vertices. The diagrams are invariant with respect to this transformation, so that it holds

$$\left.\frac{\delta \Delta[n]}{\delta n(\boldsymbol{r}, +)}\right|_{n(\boldsymbol{r}, \sigma) = n/2} = \left.\frac{\delta \Delta[n]}{\delta n(\boldsymbol{r}, -)}\right|_{n(\boldsymbol{r}, \sigma) = n/2} = \frac{\partial [\Delta(n)/V]}{\partial n} \qquad (33)$$

The consequent equalities $\ln(n_+/z) = \ln(n_-/z) = \partial[\Delta(n)/V]/\partial n$, with $\Delta(n)/V$ given by (30), lead after simple algebra to the final result

$$\frac{n^{1-\beta/4}}{z} = 2\beta^{\beta/4} \exp\left\{ \left[2C + \ln\left(\frac{\pi}{2}\right)\right]\frac{\beta}{4} + \sum_{s=1}^{\infty} d_s \beta^{L_s - N_s + 1} \right\} \qquad (34a)$$

$$= 2\beta^{\beta/4} \exp\left\{ \left[2C + \ln\left(\frac{\pi}{2}\right)\right]\frac{\beta}{4} + \frac{7}{6}\zeta(3)\left(\frac{\beta}{4}\right)^3 + \zeta(3)\left(\frac{\beta}{4}\right)^4 + O(\beta^5) \right\} \qquad (34b)$$

After Eq. (15), the length scale r_0 was set to unity. It is a simple task to show that r_0 manifests itself as the rescaling of z, $z \to r_0^{\beta/2} z$. The true dimension of the rescaled z is thus $[\text{length}]^{\beta/2 - 2}$, and Eq. (34) is dimensionally

correct. The exact solution at the collapse border $\beta = 2^{(23)}$ confirms the expected divergence $\sqrt{n}/z \to \infty$, and therefore the radius of convergence of the series $\sum_{s=1}^{\infty} \beta^{L_s - N_s + 1} d_s$ corresponds to $0 \leqslant \beta \leqslant 2$. In the next section, from the equivalence of the 2D TCP with the sine-Gordon model, the explicit form of the rhs of (34a) is found.

The internal consistency of the formalism presented in this part is confirmed by the correct reproduction of the equation of state for the pressure P of the 2D TCP,[4]

$$\beta P = n \left(1 - \frac{\beta}{4} \right) \tag{35}$$

Writing down $\ln \Xi = \beta P V$ in Eq. (3) and combining with Eq. (4), one obtains

$$\beta P = \frac{\Delta}{V} + n - n \ln \left(\frac{n}{2z} \right) \tag{36}$$

The consequent substitution of (30) and (34a) immediately leads to the exact Eq. (35).

5. MAPPING ONTO THE SINE-GORDON MODEL

It is well known that the 2D TCP is equivalent to the 2D Euclidean (classical) sin-Gordon model.[14–16] In particular, the grand partition function (1) can be turned into

$$\Xi = \frac{\int \mathscr{D}\phi \, \exp(\int d^2r \, L)}{\int \mathscr{D}\phi \, \exp[-\int d^2r \, \frac{1}{2}(\nabla\phi)^2]} \tag{37a}$$

where

$$L = -\tfrac{1}{2}(\nabla\phi)^2 + 2z \cos(\sqrt{2\pi\beta}\,\phi) \tag{37b}$$

The rescaling of the real field $\phi \to \phi/\sqrt{8\pi}$ transforms L into the form

$$L = -\frac{1}{16\pi}(\nabla\phi)^2 + 2z \cos(\bar{\beta}\phi) \tag{38a}$$

$$\bar{\beta} = \sqrt{\frac{\beta}{4}} \tag{38b}$$

which is more convenient for our purposes. If we identify one of the two spatial coordinates with an imaginary time, say $y = it$, L (38a) can be viewed as Lagrangian in the $(1 + 1)$-dimensional Minkowski space. Defining the canonical momentum conjugate to the field ϕ,

$$\Pi(x) = \frac{\partial L}{\partial \dot{\phi}} = \frac{1}{8\pi} \dot{\phi} \tag{39}$$

the Hamiltonian density of the corresponding quantum $(1 + 1)$-dimensional sine-Gordon model reads

$$H = \frac{1}{16\pi} \left[\Pi^2 + (\nabla\phi)^2 \right] - 2z \cos(\bar{\beta}\phi) \tag{40}$$

The thermodynamic properties of the 2D TCP at finite temperature may be therefore obtained from the ground-state properties of the corresponding quantum sine-Gordon model, and vice versa.

The 2D Euclidean sine-Gordon theory (38) has been intensively studied.[17–21] It can be regarded as the Gaussian Conformal FT perturbed by the relevant operator $2\cos(\bar{\beta}\phi)$. As concerns the renormalization of parameters $\bar{\beta}$ and z, only z renormalizes (multiplicatively). To give the parameter z a precise meaning, one has to fix the normalization of the field $\cos(\bar{\beta}\phi)$. Here we adopt the normalization proposed in refs. 20 and 21, which corresponds to the short-distance limit of the two-point correlation function

$$\langle \cos(\bar{\beta}\phi)(\mathbf{x}) \cos(\bar{\beta}\phi)(\mathbf{y}) \rangle \to \tfrac{1}{2} |\mathbf{x} - \mathbf{y}|^{-4\bar{\beta}^2} \qquad \text{as} \quad |\mathbf{x} - \mathbf{y}| \to 0 \tag{41}$$

Under the normalization (41), the field $\cos(\bar{\beta}\phi)$ has the scaling dimension $[\text{length}]^{-2\bar{\beta}^2}$, so that the scaling dimension of parameter z is $[\text{length}]^{2\bar{\beta}^2 - 2}$, in full agreement with (34a) $z \sim n^{1 - \beta/4}$ (the scaling dimension of the density n is $[\text{length}]^{-2}$ and $\bar{\beta}^2 = \beta/4$). The discrete symmetry of the theory (38) $\phi \to \phi + 2\pi n/\bar{\beta}$ (n integer) is broken in the region $0 < \bar{\beta} < 1$; one has to consider one of infinitely many ground states $\{|0_n\rangle\}$ characterized by $\langle \phi \rangle_n = 2\pi n/\bar{\beta}$, say that with $n = 0$. The underlying sine-Gordon model is integrable along the standard lines of the Bethe ansatz technique.[17, 18] The spectrum of particles involves solitons S, antisolitons $\bar{S}$ and soliton-antisoliton bound states $\{B_j; j = 1, 2,\dots < 1/\xi\}$,

$$\xi = \frac{\bar{\beta}^2}{1 - \bar{\beta}^2} = \frac{\beta}{4 - \beta} \tag{42}$$

with masses

$$m_j = 2M \sin\left(\frac{\pi j \xi}{2}\right) \tag{43}$$

where M is the soliton mass. The dimensionless specific grand potential ω, defined by

$$\omega = -\frac{1}{V} \ln \Xi \tag{44}$$

[and identical with the ground-state energy of the quantum $(1+1)$-dimensional sine-Gordon model minus the ground-state energy of the free-field model] was found in ref. 19, Eq. (2.67):

$$\omega = -\frac{m_1^2}{8 \sin(\pi\xi)} \tag{45}$$

Mass m_1, associated with the lightest B_1-bound state, is given by relation (43), so that

$$\omega = -\frac{M^2}{4} \tan\left(\frac{\pi\xi}{2}\right) \tag{45'}$$

Within the framework of the normalization (41), the parameter z is related to the soliton mass M as follows (see [20, Eqs. (2.12) and (4.1)])

$$z = \frac{\Gamma(\bar{\beta}^2)}{\pi \Gamma(1-\bar{\beta}^2)}\left[M \frac{\sqrt{\pi}\, \Gamma(1/2 + \xi/2)}{2\Gamma(\xi/2)} \right]^{2-2\bar{\beta}^2} \tag{46}$$

where Γ stands for the gamma function. Notice a different parameter notation in refs. 19 and 20.

Returning to the β-parameter [see formulae (38b), (42)] and applying the transformation relation between the Gamma functions $x\Gamma(x) = \Gamma(1+x)$,[26] the elimination of M from Eq. (46) implies the explicit z-dependence of ω,

$$-\omega(z, \beta) = \left(1 - \frac{\beta}{4}\right) z^{1/(1-\beta/4)} \left[2\left(\frac{\pi\beta}{8}\right)^{\beta/4} \frac{\Gamma(1-\beta/4)}{\Gamma(1+\beta/4)} \right]^{1/(1-\beta/4)}$$
$$\times \frac{\tan[\pi\beta/(2(4-\beta))]}{\pi\beta/(2(4-\beta))} \frac{\Gamma^2[1+\beta/(2(4-\beta))]}{(1/\pi)\,\Gamma^2[1/2+\beta/(2(4-\beta))]} \tag{47}$$

Inserting then $(-\omega)$ into the generating relation for the particle density (2)

$$n = z \frac{\partial(-\omega)}{\partial z} \tag{48}$$

one finally obtains

$$\frac{n^{1-\beta/4}}{z} = 2 \left(\frac{\pi\beta}{8}\right)^{\beta/4} \frac{\Gamma(1-\beta/4)}{\Gamma(1+\beta/4)}$$

$$\times \left\{ \frac{\tan[\pi\beta/(2(4-\beta))]}{\pi\beta/(2(4-\beta))} \frac{\Gamma^2[1+\beta/(2(4-\beta))]}{(1/\pi)\,\Gamma^2[1/2+\beta/(2(4-\beta))]} \right\}^{1-\beta/4} \tag{49}$$

Note that this formula is expressible in terms of the hypergeometric F-function using the identity

$$F\left(\frac{1}{2}, 2x; 1+x; 1\right) = \frac{\tan(\pi x)}{x} \frac{\Gamma^2(1+x)}{\Gamma^2(1/2+x)} \tag{50}$$

for $x = \beta/(4-\beta)/2$. The rhs of (49) predicts the stability against collapse for $\beta < 2$ and the divergency of n at $\beta \to 2$, as it should be. Its expansion around $\beta = 0$ up to the β^4 term, accomplished e.g., by using the symbolic computer language *Mathematica*, turns out to be identical to our previous result (34b). This is the strong evidence that formula (49) reflects adequately the diagrammatic series $\sum_{s=1}^{\infty} \beta^{L_s-N_s+1} d_s$ in (34a), and represents the exact density–fugacity relationship for the 2D TCP.

We would like to stress that there were attempts in the past to get the thermodynamics of the 2D TCP from the mapping onto the sine-Gordon theory, but they failed because of an inadequate transfer of the information. We mention one earlier work,[27] the result of which was proved to be wrong in the report.[28] The crucial point here is that the dependence of the parameter z on the soliton mass M, relation (46), was determined within the field normalization (41), the last being consistent with the formalism of the renormalized Mayer expansion (Sections 3 and 4).

6. THERMODYNAMICS OF THE 2D TCP

The simplest way to get the thermodynamics of the 2D TCP is to pass from the grandcanonical to the canonical ensemble via the Legendre transformation

$$F(T, N) = \Omega + \mu N \tag{51}$$

where

$$\Omega = k_B T \, \omega(\beta, n) \, V \tag{52a}$$

$$-\omega(\beta, n) = \left(1 - \frac{\beta}{4}\right) n \tag{52b}$$

and

$$\mu(\beta, n) = k_B T \ln z(\beta, n) \tag{53}$$

The dimensionless specific free energy f, defined by $f = F/(N k_B T)$, is then written as

$$
\begin{aligned}
f(\beta, n) = & -\left(1 - \frac{\beta}{4}\right) + \left(1 - \frac{\beta}{4}\right) \ln n - \ln\left[2\left(\frac{\pi\beta}{8}\right)^{\beta/4} \frac{\Gamma(1 - \beta/4)}{\Gamma(1 + \beta/4)} \right] \\
& -\left(1 - \frac{\beta}{4}\right) \ln \left\{ \frac{\tan[\pi\beta/(2(4-\beta))]}{\pi\beta/(2(4-\beta))} \frac{\Gamma^2[1 + \beta/(2(4-\beta))]}{(1/\pi) \, \Gamma^2[1/2 + \beta/(2(4-\beta))]} \right\}
\end{aligned}
\tag{54}
$$

According to the elementary thermodynamics, the (excess) internal energy per particle, $u = \langle E \rangle / N$, and the (excess) specific heat at constant volume per particle, $c_V = C_V/N$, are given by

$$u = \frac{\partial}{\partial \beta} f(\beta, n) \tag{55a}$$

$$\frac{c_V}{k_B} = -\beta^2 \frac{\partial^2}{\partial \beta^2} f(\beta, n) \tag{55b}$$

For c_V, one gets explicitly

$$
\begin{aligned}
\frac{c_V}{k_B} = & \frac{\beta}{4} + \frac{4}{4 - \beta} + \frac{\beta^2}{16} \left[\psi^{(1)}\left(1 - \frac{\beta}{4}\right) - \psi^{(1)}\left(1 + \frac{\beta}{4}\right) \right] \\
& - 2 \frac{\beta^2}{(4-\beta)^3} \left[\psi^{(1)}\left(\frac{2}{4-\beta}\right) - \psi^{(1)}\left(\frac{8-\beta}{8-2\beta}\right) \right] \\
& - \frac{4\pi^2\beta^2}{(4-\beta)^3} \frac{\cos(\pi\beta/(4-\beta))}{\sin^2(\pi\beta/(4-\beta))}
\end{aligned}
\tag{56}
$$

The series representation of the derivatives of the psi function is given by the formula[26]

$$\psi^{(n)}(x) = (-1)^{n+1} n! \sum_{j=0}^{\infty} \frac{1}{(x+j)^{n+1}} \tag{57}$$

The density-independent c_V/k_B has a regular high-temperature expansion in powers of β:

$$\frac{c_V}{k_B} = \frac{\beta}{4} + \frac{7}{64} \zeta(3) \beta^3 + \frac{3}{64} \zeta(3) \beta^4 + O(\beta^5) \tag{58}$$

The expansion of c_V/k_B around the collapse $\beta = 2$ point results in the Laurent series

$$\frac{c_V}{k_B} = \frac{2}{(2-\beta)^2} - \frac{3}{(2-\beta)} + \frac{3}{2} + \frac{1}{4} [17\zeta(3) - 1](2-\beta) + O[(2-\beta)^2] \tag{59}$$

The leading term coincides exactly with the conjecture of Hauge and Hemmer[4] [see their Eq. (34)] based on an independent-pair approximation. The Monte-Carlo simulation estimates of u and c_V[29] agree well with our exact results for the coupling strengths $\beta = 1$ and $\beta = 1.5$, however, they become inaccurate near the collapse border (namely at $\beta = 1.8$).

   In conclusion, it was shown that the renormalized reformulation of the Mayer expansion in statistical mechanics of the 2D TCP produces a direct link to techniques in the related QFT. The obtained results might represent a step towards the complete integrability of the 2D TCP on higher correlation levels. A further exploration of the intimate relationship to recent studies of the sine-Gordon model is needed.

APPENDIX

   We aim at proving the relations

$$\int \frac{d^2r}{2\pi} K_0^4(\mathbf{r}) = \frac{7}{8} \zeta(3) \tag{A1a}$$

$$\int \frac{d^2r_1}{2\pi} \int \frac{d^2r_2}{2\pi} K_0^2(\mathbf{r}_1) K_0^2(\mathbf{r}_2) K_0^2(\mathbf{r}_1 - \mathbf{r}_2) = \frac{3}{16} \zeta(3) \tag{A1b}$$

where K_0 is the modified Bessel function of second kind and ζ is the Riemann's zeta function. The Fourier component of $K_0^2(\boldsymbol{r})$, denoted as $G(\boldsymbol{k})$, can be expressed in 2D as follows

$$\begin{aligned}
G(\boldsymbol{k}) &= \int \frac{\mathrm{d}^2 r}{2\pi} \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r}) \, K_0^2(\boldsymbol{r}) \\
&= \int_0^\infty \mathrm{d}r \, r J_0(kr) \, K_0^2(r) = \frac{\ln[k/2 + \sqrt{1 + (k/2)^2}]}{k\sqrt{1 + (k/2)^2}}
\end{aligned} \tag{A2}$$

where J_0 is the ordinary Bessel function. Simple algebra yields

$$\int \frac{\mathrm{d}\boldsymbol{r}}{2\pi} K_0^2(\boldsymbol{r}) \, K_0^2(\boldsymbol{r}) = \int_0^\infty \mathrm{d}k \, k G^2(k) \tag{A3a}$$

$$\int \frac{\mathrm{d}\boldsymbol{r}_1}{2\pi} \int \frac{\mathrm{d}\boldsymbol{r}_2}{2\pi} K_0^2(\boldsymbol{r}_1) \, K_0^2(\boldsymbol{r}_2) \, K_0^2(\boldsymbol{r}_1 - \boldsymbol{r}_2) = \int_0^\infty \mathrm{d}k \, k G^3(k) \tag{A3b}$$

Under the change of variable $k/2 = \sinh(t/2)$ relations (A3a) and (A3b) become

$$\int_0^\infty \mathrm{d}k \, k G^2(k) = \frac{1}{2} \int_0^\infty \mathrm{d}t \, \frac{t^2}{2\sinh t} \tag{A4a}$$

$$\int_0^\infty \mathrm{d}k \, k G^3(k) = \frac{1}{2} \int_0^\infty \mathrm{d}t \, \frac{t^3}{(2\sinh t)^2} \tag{A4b}$$

respectively. Let us expand

$$\frac{1}{2\sinh t} = \sum_{j=1}^\infty \mathrm{e}^{-(2j-1)\,t} \tag{A5a}$$

$$\frac{1}{(2\sinh t)^2} = \sum_{j=1}^\infty j\mathrm{e}^{-2jt} \tag{A5b}$$

Integrating each term one obtains

$$\frac{1}{2} \int_0^\infty \mathrm{d}t \, \frac{t^2}{2\sinh t} = \frac{2!}{2} \sum_{j=1}^\infty \frac{1}{(2j-1)^3} = \left(1 - \frac{1}{8}\right)\zeta(3) \tag{A6a}$$

$$\frac{1}{2} \int_0^\infty \mathrm{d}t \, \frac{t^3}{(2\sinh t)^2} = \frac{3!}{2^5} \sum_{j=1}^\infty \frac{1}{j^3} = \frac{3}{16}\zeta(3) \tag{A6b}$$

what proves (A1a) and (A1b).

ACKNOWLEDGMENTS

REFERENCES

 1. Ph. A. Martin, *Rev. Mod. Phys.* **60**:1075 (1988).
 2. F. H. Stillinger and R. Lovett, *J. Chem. Phys.* **48**:3858 (1968).
 3. F. H. Stillinger and R. Lovett, *J. Chem. Phys.* **49**:1991 (1968).
 4. E. H. Hauge and P. C. Hemmer, *Phys. Norvegica* **5**:209 (1971).
 5. P. Vieillefosse and J. P. Hansen, *Phys. Rev. A* **12**:1106 (1975).
 6. B. Jancovici, A Sum Rule for the 2d TCP, LPT Orsay 99-59, cond-mat/9907365, to appear in *J. Stat. Phys.*
 7. B. Jancovici, P. Kalinay, and L. Šamaj, *Physica A* **279**:260 (2000).
 8. C. Deutsch and M. Lavaud, *Phys. Rev. A* **9**:2598 (1974).
 9. P. Kalinay, P. Markoš, L. Šamaj, and I. Travěnec, *J. Stat. Phys.* **98**:639 (2000).
10. E. E. Salpeter, *Ann. Phys. (N.Y.)* **5**:183 (1958).
11. P. J. Forrester, *J. Stat. Phys.* **63**:491 (1991).
12. B. Jancovici, G. Manificat, and C. Pisani, *J. Stat. Phys.* **76** (1994) 307.
13. J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon, Oxford, 1989).
14. S. F. Edwards, *Philos. Mag.* **4**:1171 (1959).
15. D. Brydges and P. Federbush, *Commun. Math. Phys.* **73**:197 (1980).
16. P. Minnhagen, *Rev. Mod. Phys.* **59**:1001 (1987).
17. L. D. Faddeev and L. A. Takhtajan, *Teor. Mat. Fiz.* **21**:160 (1974).
18. A. B. Zamolodchikov and Al. B. Zamolodchikov, *Ann. Phys. (N.Y.)* **120**:253 (1979).
19. C. Destri and H. de Vega, *Nucl. Phys. B* **358**:251 (1991).
20. Al. B. Zamolodchikov, *Int. J. Mod. Phys. A* **10**:1125 (1995).
21. S. Lukyanov and Al. B. Zamolodchikov, *Nucl. Phys. B* **493**:571 (1997).
22. M. Gaudin, *J. Phys. (France)* **46**:1027 (1985).
23. F. Cornu and B. Jancovici, *J. Chem. Phys.* **90**:2444 (1989).
24. K. Wilson, *Phys. Rev.* **179**:1499 (1969).
25. J. P. Hansen and I. R. McDonald, *Theory of Simple Liquids*, 2nd edn. (Academic Press, London, 1990).
26. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, 5th edn. (Academic Press, London, 1994).
27. A. M. Saglio, *Nucl. Phys. B* [FS6] **210**:369 (1982).
28. K. Olaussen, *Specific Heat of the 2D Coulomb Gas* (1989), unpublished report.
29. J. P. Hansen and P. Viot, *J. Stat. Phys.* **38**:823 (1985).